

WHICH ARE BIGGER?

The natural numbers $\{0,1,2,3,\dots\}$

The prime numbers $\{2,3,5,7, 11,\dots\}$

The odd numbers $\{3,5,7,9, 11,\dots\}$

The rational numbers $\{m/n \text{ for integer } m,n\} = \mathbb{Q}$

2^ω = number of heads/tails sequences for an infinite number of coin flips

The real numbers $\{1, 2/3, \sqrt{2}, e, \pi, \text{etc.}\} = \mathbb{R}$

The naturals, primes, and odd numbers are the same size - each can be listed where any given member will appear in a finite time. These are called countably infinite

0 1 2 3 4 5 6 7 8 9 10 ... = nats

2 3 5 7 11 13 17 19 21 23 29 ... = primes

3 5 7 9 11 13 15 17 19 21 23 ... = odds

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The rationals *can* be listed, but not in the 'standard' order of less than

Here is an ordering of all the positive rationals

$1/1$ $2/1$ $3/1$ $4/1$ $5/1$ $6/1$ $7/1$ $8/1$... = nats

$1/2$ $2/2$ $3/2$ $4/2$ $5/2$ $6/2$ $7/2$ $8/2$...

$1/3$ $2/3$ $3/3$ $4/3$ $5/3$ $6/3$ $7/3$ $8/3$

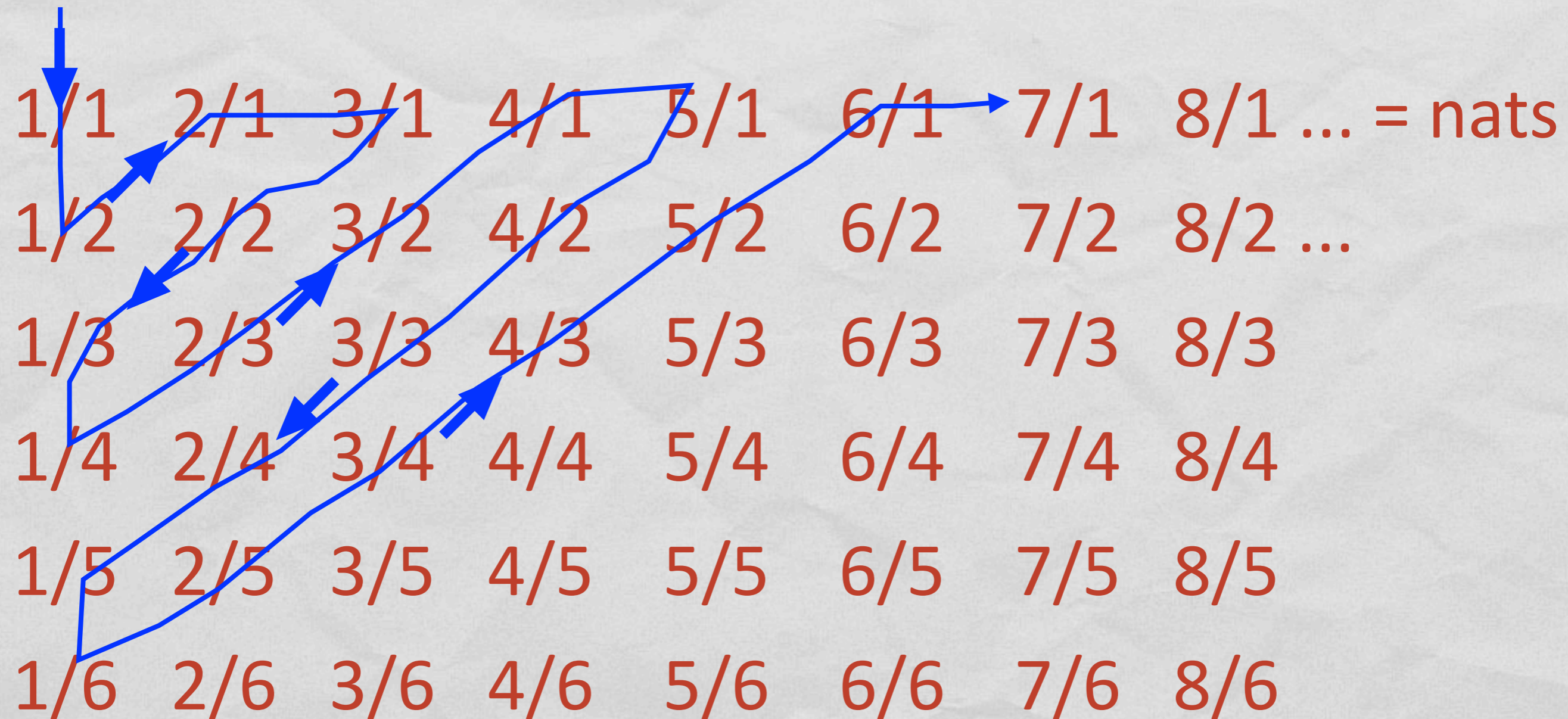
$1/4$ $2/4$ $3/4$ $4/4$ $5/4$ $6/4$ $7/4$ $8/4$

$1/5$ $2/5$ $3/5$ $4/5$ $5/5$ $6/5$ $7/5$ $8/5$

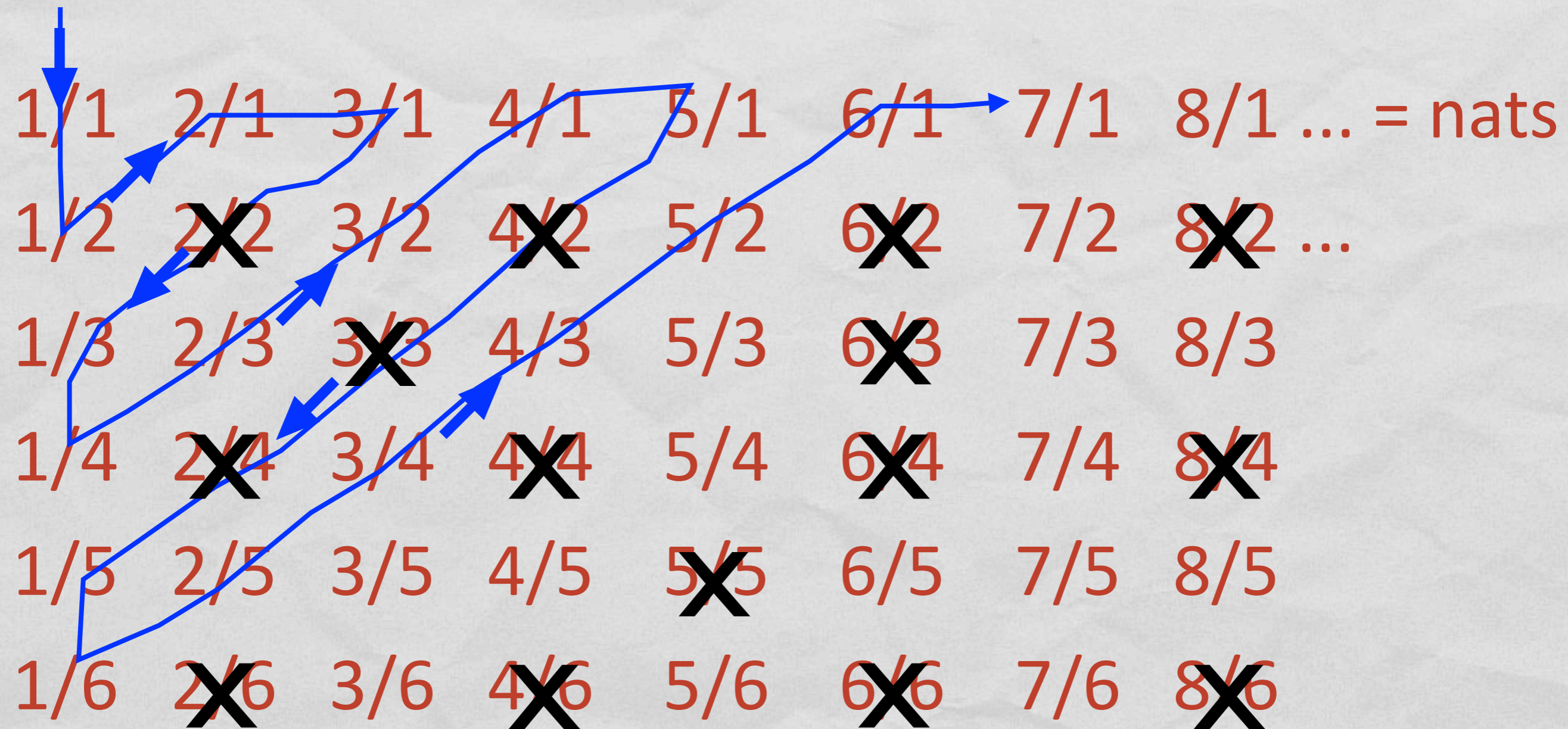
$1/6$ $2/6$ $3/6$ $4/6$ $5/6$ $6/6$ $7/6$ $8/6$

...an infinite number of rows of infinite length

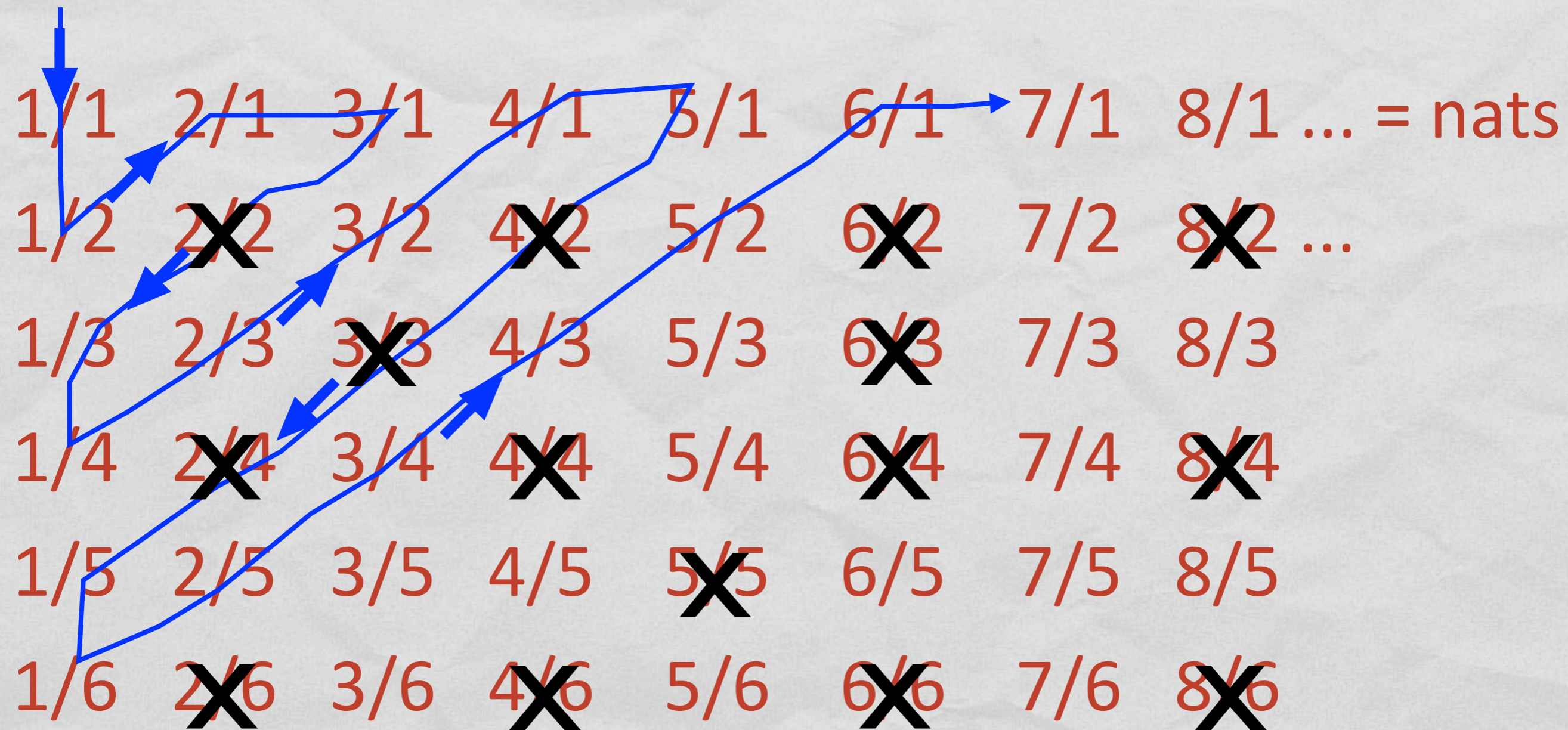
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Each rational has a specific place in the sequence

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First, note that every real number can be represented by an infinite expansion of digits after a decimal. $1/3 = .333333\dots$, but also, $1/2 = 0.500000000\dots$

Here is a purported list

.0123456789100101010102929292.... = real₁

.0513451589150115011502511111.... = real₂

.2222222222222222222222222222.... = real₃

.1313131313131313131313131313.... = real₄

.7774447774447774447774447774.... = real₅

..... = real₆

.....

..... etc.

Here is a purported list

digit 1 digit 10 digit 20



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Here is a number not on the list:

.16345... where the n^{th} digit is the n^{th} digit of $\text{real}_n + 1$. It is at least one digit different than any number on the list

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(little quirks like the fact that $.0999999999\dots$
= $.1000000000\dots$ can be worked out)

WE CAN ALWAYS GET BIGGER

- The reals are not the biggest set of course. Take all the subsets of the reals. This set is bigger (size $2^{2^{\omega}}$ to be exact).
- For any set, the powerset is bigger. So there is no biggest infinite number. Just like there is no biggest finite number.

SET THEORY

Wednesday, 1 December

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and yes, $\vdash \forall x (x < 2^x)$ even for infinite x

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From these two we can prove $U = \wp(U)$

-- but we just showed that $\vdash |\wp(U)| < |U|$

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$\vdash \neg \exists y \forall x [x \in y \leftrightarrow x \notin x]$ by Russell's argument

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So from a set z , you can shrink it down.

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+Replacement, Foundation, Choice make ZFC Set Theory

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- Manifolds, Vector Spaces, Probability Spaces, and anything else you think of are sets.

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 - Relations are sets of ordered pairs $R(a, b)$ means $\langle x, y \rangle \in R$

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 - View 2: All mathematical objects can be modeled by sets and by proving things about the relevant sets, you can thereby understand these objects.
 - View 3: (My view) Set Theory is powerful and helps make precise lots of important notions. But it can't cover all of mathematics and definitely not all of logic.

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Recursion Theory (Computability Theory) - is there a general algorithm for determining whether a given kind of equation has integer roots?