Which are Bigger?

The natural numbers {0,1,2,3....}

The prime numbers {2,3,5,7, 11....}

The odd numbers {3,5,7,9, 11....}

The rational numbers {m/n for integer m,n} = $\mathbb{Q}$

2^{ω} = number of heads/tails sequences for an infinite number of coin flips

The real numbers {1, 2/3, √2, e, π, etc.} = $\mathbb{R}$

The naturals, primes, and odd numbers are the same size - each can be listed where any given member will appear in a finite time. These are called <u>countably infinite</u>

0	1	2	3	4	5	6	7	8	9	10 ...	= nats
2	3	5	7	11	13	17	19	21	23	29 ...	= primes
3	5	7	9	11	13	15	17	19	21	23 ...	= odds

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The rationals *can* be listed, but not in the 'standard' order of less than

Here is an ordering of all the positive rationals

1/1	2/1	3/1	4/1	5/1	6/1	7/1	8/1 ... = nats
1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2 ...
1/3	2/3	3/3	4/3	5/3	6/3	7/3	8/3
1/4	2/4	3/4	4/4	5/4	6/4	7/4	8/4
1/5	2/5	3/5	4/5	5/5	6/5	7/5	8/5
1/6	2/6	3/6	4/6	5/6	6/6	7/6	8/6

...an infinite number of rows of infinite length

Follow the blue arrows zigzagging through the whole set

1/1	2/1	3/1	4/1	5/1	6/1	7/1	8/1	... = nats
1/2	2/2	3/2	4/2	5/2	6/2	7/2	8/2	...
1/3	2/3	3/3	4/3	5/3	6/3	7/3	8/3	
1/4	2/4	3/4	4/4	5/4	6/4	7/4	8/4	
1/5	2/5	3/5	4/5	5/5	6/5	7/5	8/5	
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1/2	~~2/2~~	3/2	~~4/2~~	5/2	~~6/2~~	7/2	~~8/2~~	...
1/3	2/3	~~3/3~~	4/3	5/3	~~6/3~~	7/3	8/3	
1/4	~~2/4~~	3/4	~~4/4~~	5/4	~~6/4~~	7/4	~~8/4~~	
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Each rational has a specific place in the sequence

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First, note that every real number can be represented by an infinite expansion of digits after a decimal. 1/3 = .333333..., but also, 1/2 = 0.500000000...

Here is a purported list

.0123456789100101010102929292.... = $real_1$
.0513451589150115011502511111.... = $real_2$
.2222222222222222222222222222.... = $real_3$
.1313131313131313131313131313.... = $real_4$
.7774447774447774447774447774.... = $real_5$
...... = $real_6$
......
...... etc.

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digit 1 digit 10 digit 20

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Here is a number not on the list:
.16345... where the n^{th} digit is the n^{th} digit of $real_n + 1$. It is at least one digit different than any number on the list

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(little quirks like the fact that .099999999...
= .100000000... can be worked out)

WE CAN ALWAYS GET BIGGER

- The reals are not the biggest set of course. Take all the subsets of the reals. This set is bigger (size $2^{\wedge}2^{\omega}$ to be exact).
- For any set, the powerset is bigger. So there is no biggest infinite number. Just like there is no biggest finite number.

Set Theory

Wednesday, 1 December

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and yes, $\vdash \forall x\ (x < 2^x)$ even for infinite x

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From these two we can prove $U = \wp(U)$
-- but we just showed that $\vdash |\wp(U)| < |U|$

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$R \in R \leftrightarrow R \notin R$

$\vdash \neg \exists y \forall x[x \in y \leftrightarrow x \notin x]$ by Russell's argument

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So from a set z, you can shrink it down.

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- Axiom of Infinity (the natural numbers are a set)
- +Replacement, Foundation, Choice make ZFC Set Theory

Set Theory as the Foundations of Mathematics

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    - Algebra - groups, rings, fields, etc. are just sets that obey certain axioms
    - Analysis - the real numbers are [the] complete Archimedean field. They are sets that satisfy certain axioms.
    - Manifolds, Vector Spaces, Probability Spaces, and anything else you think of are sets.

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  - Relations are sets of ordered pairs R(a,b) means $<x,y> \in R$

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  - View 2: All mathematical objects can be modeled by sets and by proving things about the relevant sets, you can thereby understand these objects.
  - View 3: (My view) Set Theory is powerful and helps make precise lots of important notions. But it can't cover all of mathematics and definitely not all of logic.

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Recursion Theory (Computability Theory) - is there a general algorithm for determining whether a given kind of equation has integer roots?